10.1 Differential Equations

A differential Equation is an equation containing derivatives of an unknown function $y = f(t)$.

Ex/

$$y' = ky \qquad \text{has solution } y = Ce^{kt}$$

For Differential Equations our unknown is a function rather than a number.

How to "verify" a solution?

ex/

$$y'' + 9y = t$$

Is the solution

Ⓐ $y = \frac{1}{9}t + \sin 2t$

or

Ⓑ $y = \frac{1}{9}t + \sin 3t$

or

Ⓒ $y = \frac{1}{9}t + \cos 3t$

Ⓐ $y = \frac{1}{9}t + \sin 2t$

$y' = \frac{1}{9} + 2\cos 2t$

$y'' = -4\sin 2t$

so

$$-4\sin 2t + 9\left(\tfrac{1}{9}t + \sin 2t\right) \overset{?}{=} t$$

$$-4\sin 2t + t + 9\sin 2t \overset{?}{=} t$$

$$t + 5\sin 2t \neq t$$

(B) $y = \frac{1}{9}t + \sin 3t$

$y' = \frac{1}{9} + 3\cos 3t$

$y'' = -9\sin 3t$

$$-9\sin 3t + 9\left(\tfrac{1}{9}t + \sin 3t\right) \stackrel{?}{=} t$$

$$t \stackrel{\checkmark}{=} t$$

(C) $y = \frac{1}{9}t + \cos 3t$

$y' = \frac{1}{9} - 3\sin 3t$

$y'' = -9\cos 3t$

$$-9\cos 3t + 9\left(\tfrac{1}{9}t + \cos 3t\right) \stackrel{?}{=} t$$

$$t \stackrel{\checkmark}{=} t$$

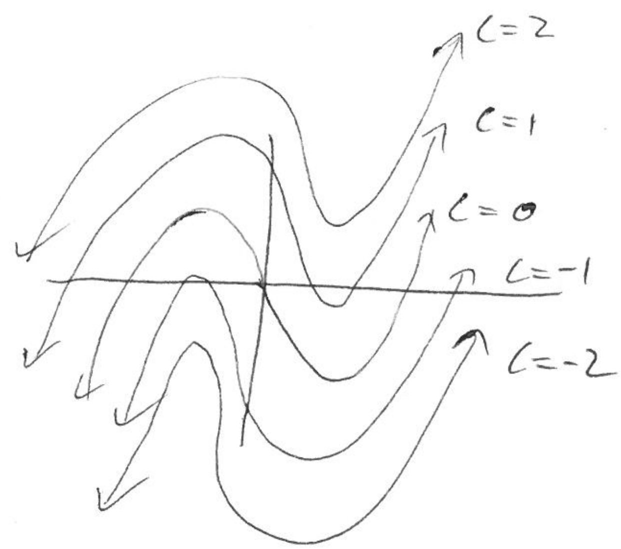
Simple kind of differential equation:

ex

$$y' = 3t^2 - 4$$

$$y = \int 3t^2 - 4\, dt$$

$$y = t^3 - 4t + \underline{C}$$



The solution gives us a family of solution curves.

Solving a differential equation gives us all possible solutions.

ex/

$y' = 3y$ has a solution $y = 2e^{3t}$

but general solution $y = Ce^{3t}$

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ex/ $y'' = 6t$ $\quad y' = \int 6t\,dt = 3t^2 + C \quad y = \int 3t^2 + C\,dt = t^3 + Ct + D$

Particular solutions to a differential equation that satisfies some condition $y(0) = t_0$ is called an initial-value problem

ex/ Solve the IVP

$y' = 2t + 5 \qquad y(0) = 5$

$\int 2t + 5\,dt = t^2 + 5t + C$

$y = t^2 + 5t + C$

$y(0) = 5$ so

$5 = 0^2 + 5(0) + C$

$C = 5$ $\quad \cancel{y = t^2 + 5t} \quad \boxed{y = t^2 + 5t + 5}$

ex Solve the IVP

$y' = 3y \qquad y(0) = 8$

$y = Ce^{3t}$

$y(0) = 8$ so

$8 = Ce^{3(0)}$

$C = 8$

$\boxed{y = 8e^{3t}}$

Often useful to find constant solutions. $y = a$ so $y' = 0$

ex $y' = 3y - 12$

$0 = 3a - 12$

$3a = 12$

$a = 4$

when $y = 4$
$y' = 0$

ex what are the constant solutions to

$y' = y^2 + 8y + 15$?

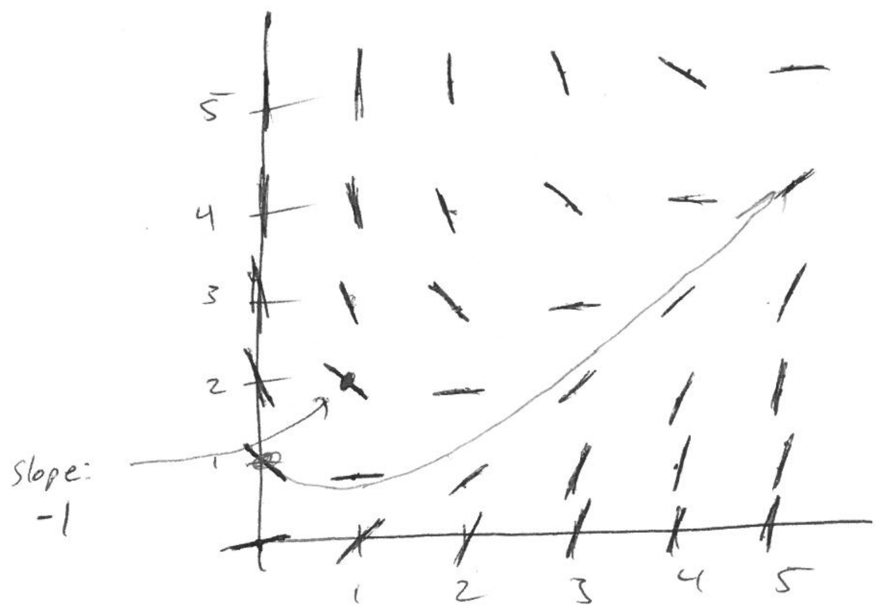
Slope fields:

with a differential equation
such as $y' = t - y$
we can understand how y behaves by sketching what the slope of y is at points (t, y)

at $(1,2)$
$y' = 1 - 2 = -1$

at $(1,1)$
$y' = 0$

at $(2,1)$
$y' = 2 - 1 = 1$



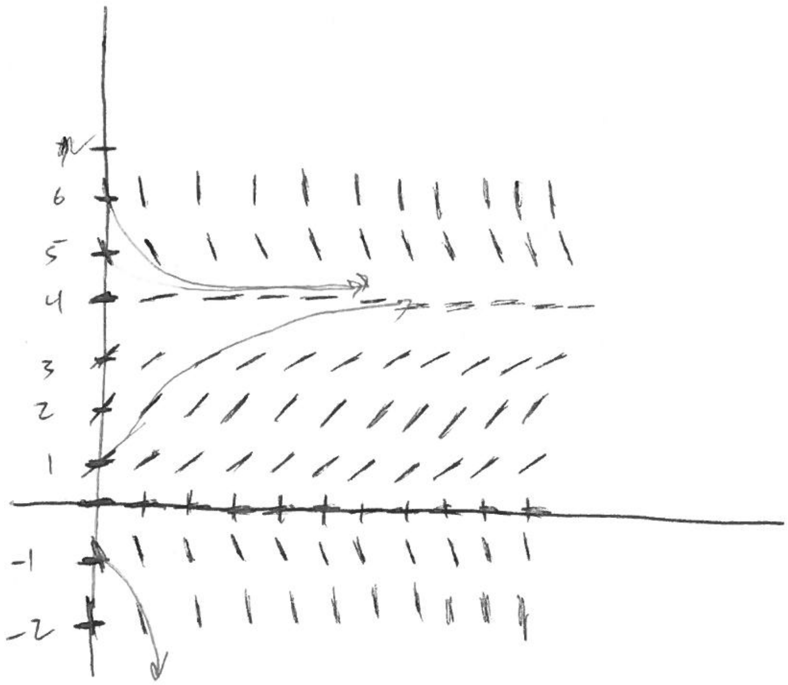
starting at $(0,1)$ we "follow" the slope field.

ex/ Sketch a slope field for

$y' = \frac{1}{4} y (4 - y)$ on integer points
$0 \le t \le 5$
$-2 \le y \le 6$

and sketch y given initial conditions $(0,1)$, $(0,-1)$, $(0,5)$

note: only depends on y-values



$y=0$
$y' = 0$

$y=1$ $y' = \frac{3}{4}$

$y=2$ $y' = 1$

$y=3$ $y' = \frac{3}{4}$

$y=4$ $y' = 0$

$y=5$ $y' = -\frac{5}{4}$

$y=6$ $y' = -3$

$y=-1$ $y' = -\frac{5}{4}$

$y=-2$ $y' = -3$